Verification and validation of least-squares fictitious domain method with finite-hp element approximation

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ABSTRACT

The proposed numerical method, based on the fictitious domain approach, has been conceived for the solution of differential problems defined on domain changing in time and space, i.e. in general structural elastic problems, fluid dynamics problems with moving rigid bodies, shape optimization problems, differential equations defined on stochastic domain, and so on. This means the same problem is solved on different domains. Unlike the usual approach, based on the boundary variation technique where a sequence of domains is considered, according to fictitious domain approach the computational domain is not the same as the physical domain of the problem, but it contains that one (Fig.1). Hence when the physical domain changes the computational domain does not change with evident advantages.

Several variants of the fictitious domain method exist: the basic idea is to extend the operator and the domain into a larger simple shaped domain. The most important ways to do this are algebraic and functional analytic approaches. As argued in [1] the algebraic fictitious domain methods can be rather efficient, but typically they are restricted to quite a narrow class of problems [2]. More flexibility and better efficiency can be obtained by using a functional analytic approach [3] where the use of constraints ensures that the solution of the extended problem coincides with the solution of the original problem. In our implementation we enforce constraints by Lagrange multipliers according to the boundary lagrangian technique [4,5].

The physical aspects of the problem can always be stated in a least-squares variational principle form, which specifies a scalar quantity, the functional \( J \), defined by an integral form. The solution to the continuum problem is a function which makes \( J \) stationary with respect to small changes of the unknown function; thus, for a solution to the continuum problem, it is \( \delta J = 0 \). To implement the fictitious domain approach we have to extend the mathematical operator in the functional \( J \) and the physical domain \( \Omega \) into a larger simple shaped domain \( \Pi \) and we have to constrain the functional on the immersed boundary \( \Gamma \) (Fig.2). To enforce the functional on the immersed boundary Lagrange multipliers are introduced, so that the problem is now equivalent to find the stationary point of the Lagrangian function:

\[
\Lambda = \frac{1}{2} \int_{\Pi} F(\phi, \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \ldots, x, y, \ldots) \ d\pi + \int_{\Gamma} \lambda(x, y, \ldots) E(\phi, \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \ldots, x, y, \ldots) d\gamma
\]  

(1)
where $F$ and $E$ are specified operators, $\phi$ is the unknown function dependent on variables $(x,y,\ldots)$ and $\lambda$ is the unknown multiplier function of position. To solve the so formulated saddle-point problem, we discretize it into spectral or finite-hp elements.

The novelty of the proposed method, compared to already known formulations, is just this one. In fact although many methods have been proposed in a finite element, finite difference or finite volume context, only few publications exist on the performance of non-matching approaches for high-order discretizations.

Here we propose the finite-hp element discretization. The finite-hp approximation is based on higher order functions, locally defined over the elements of domain. The advantage of such kind of method, in comparison with traditional finite element method, is its exponential convergence property with the increasing of polynomial order $p$.

In this work the least squares fictitious domain formulation based on finite-hp element approximation is verified by several two-dimensional analytical test-cases with increasing difficulty. We consider the solution of the Poisson equation on complex geometry, the solution of steady conduction-advection equation, the solution of Wannier flow (Stokes equations) and the solution of Kovasznay flow (Navier-Stokes equations) (Fig.3). In particular the spectral convergence of the numerical model is demonstrated.

(a)                                                       (b)                                                        (c)                                                        (d)

Fig. 3 Analytical test-cases to verify the numerical model: (a) Poisson equation, (b) conduction-advection equation, (c) Wannier flow, (d) Kovasznay flow.

The methodology is also validated by numerical and experimental data found in literature for the flow past a circular cylinder (Fig.4).

(a)                                                       (b)

Fig. 4 Test-case for the numerical model validation: flow past a circular cylinder.

References:


