Development of an accurate method for advection problems with embedded moving-boundaries

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Academy colloquium on IBMs  
15 June 2009, Amsterdam
1. Introduction
2. Spatial discretization
3. Temporal discretization
4. Results
5. Conclusion
future: computation of fluid flows around complex moving bodies
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present: numerical analysis for model problems: development of a monotone, higher-order accurate and efficient method for convection, using a fixed-grid FVM
advection eqn: $c_t + (uc)_x = 0$, $u = \text{constant} > 0$

initial conditions:

$$c(x, 0) = \begin{cases} 
0, & \text{if } x_1 \leq x \leq x_2 \\
1, & \text{elsewhere} 
\end{cases}$$

with periodic boundary condition
advection eqn: \( c_t + (uc)_x = 0, \ u = \text{constant} > 0 \)

initial conditions:

\[
c(x, 0) = \begin{cases} 
0, & \text{if } x_1 \leq x \leq x_2 \\
\frac{1}{2}(1 - \cos(2\pi x)), & \text{elsewhere}
\end{cases}
\]
domain of unit length, \( x \in [0, 1] \)
divided into \( N \) uniform cells: \( h = 1/N \)
cell-averaged discrete solution
domain of unit length, $x \in [0, 1]$

divided into $N$ uniform cells: $h = 1/N$

cell-averaged discrete solution

semi-discrete equation:

$$h \frac{dc_i}{dt} + u \left( c_{i+1/2}(t) - c_{i-1/2}(t) \right) = 0$$
cell-face state approximation:

- first-order accurate upwind: \( c_{i+\frac{1}{2}} = c_i \)
cell-face state approximation:

- first-order accurate upwind: \( c_{i+\frac{1}{2}} = c_i \)
- high-order accurate scheme
  - unlimited \( \kappa \)-scheme, \( \kappa \in [-1, 1] \):
    \[
    c_{i+\frac{1}{2}} = c_i + \frac{1 + \kappa}{4} (c_{i+1} - c_i) + \frac{1 - \kappa}{4} (c_i - c_{i-1})
    \]
  - limited \( \kappa \)-scheme: \( c_{i+\frac{1}{2}} = c_i + \frac{1}{2} \phi(r_{i+\frac{1}{2}})(c_i - c_{i-1}) \)
    \[
    r_{i+\frac{1}{2}} = \frac{c_{i+1} - c_i}{c_i - c_{i-1}}
    \]
standard limiter

limiter for $\kappa = 1/3$

$$\phi(r) = \max(0, \min(2r, \min(\frac{1}{3} + \frac{2}{3}r, 2)))$$
standard FVM results

- no embedded boundary-conditions imposed yet: pure capturing
- $u = 1$, $x_1 = \frac{1}{3}$, $x_2 = \frac{2}{3}$, $T = 1$ (one full-period)
- very small time steps
- number of cells: $N = 20$ (top) and $N = 40$ (bottom)
exact discrete (−), first-order upwind (−), unlimited $\kappa = \frac{1}{3}$ (−), limited $\kappa = \frac{1}{3}$ (−)
embedded-boundary conditions:

- discontinuities in initial solutions model infinitely thin bodies going with flow
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- discontinuities in initial solutions model infinitely thin bodies going with flow
- solution values on left and right sides of EB: $c_{EB}^l$ and $c_{EB}^r$
- user-specified values
- embedded in fixed-grid fluxes
EB in cell $i$ at time $t^n$ and affected cell-face states

\[
\begin{align*}
  c_{i-1/2}^l &
  \\
  c_{i+1/2}^r &
  \\
  c_{i+3/2}^l &
\end{align*}
\]

EB-position relative to $x_{i-1/2}$ is $\beta h$: $\beta = \frac{x_{EB} - x_{i-1/2}}{\frac{1}{2} h}$, $\beta \in [0, 1]$
EB in cell $i$ at time $t^n$ and affected cell-face states

$\mathbf{c}^E_B$ | $\mathbf{c}^R_B$
---|---
$c_{i-\frac{1}{2}}$, $c_{i+\frac{1}{2}}$, $c_{i+\frac{3}{2}}$

EB-position relative to $x_{i-\frac{1}{2}}$ is $\beta h$: $\beta = \frac{x_{EB} - x_{i-\frac{1}{2}}}{h}$, $\beta \in [0, 1]$

no information flow across EB

cell-face states affected in case of 3-point upwind-biased interpolation: $c_{i-\frac{1}{2}}(\mathbf{c}^E_B, \cdot)$, $c_{i+\frac{1}{2}}(\mathbf{c}^E_B, \cdot)$, $c_{i+\frac{3}{2}}(\mathbf{c}^E_B, \cdot)$
unlimited, higher-order, EB-affected, cell-face states

\[ c_{i-\frac{1}{2}} = c_{i-1} + \frac{1}{1+2\beta} \left( \frac{1+\kappa_{i-\frac{1}{2}}}{2} (c_{EB}^l - c_{i-1}) + \frac{1-\kappa_{i-\frac{1}{2}}}{4} (c_{i-1} - c_{i-2}) \right) \]

\[ c_{i+\frac{1}{2}} = c_{EB}^r + \frac{2-2\beta}{3-2\beta} (c_{i+1} - c_{EB}^r) \]

\[ c_{i+\frac{3}{2}} = c_{i+1} + \frac{1}{4} \left( \frac{1+\kappa_{i+\frac{3}{2}}}{2} (c_{i+2} - c_{i+1}) + \frac{1-\kappa_{i+\frac{3}{2}}}{2} (c_{i+1} - c_{EB}^r) \right) \]
unlimited, higher-order, EB-affected, cell-face states

\[ \begin{align*}
    c_{i - \frac{1}{2}} &= c_{i-1} + \frac{1}{1+2\beta} \left( \frac{1 + \kappa_{i-\frac{1}{2}}}{2} (c_{EB}^l - c_{i-1}) + \frac{1 - \kappa_{i-\frac{1}{2}}}{4} (c_{i-1} - c_{i-2}) \right) \\
    c_{i + \frac{1}{2}} &= c_{EB}^r + \frac{2 - 2\beta}{3 - 2\beta} (c_{i+1} - c_{EB}^r) \\
    c_{i + \frac{3}{2}} &= c_{i+1} + \frac{1 + \kappa_{i+\frac{3}{2}}}{4} (c_{i+2} - c_{i+1}) + \frac{1 - \kappa_{i+\frac{3}{2}}}{2} (c_{i+1} - c_{EB}^r) \\
\end{align*} \]

\( \kappa_{i - \frac{1}{2}} \) and \( \kappa_{i + \frac{3}{2}} \) to be optimised for net fluxes

away from EB, standard \( \kappa = \frac{1}{3} \) scheme is used
optimum $\kappa$ values

$$\kappa_{i-\frac{1}{2}} = \frac{7 - 6\beta}{9 + 6\beta}$$
optimum $\kappa$ values

\[
\kappa_{i-\frac{1}{2}} = \frac{7 - 6\beta}{9 + 6\beta}
\]

\[
\kappa_{i+\frac{3}{2}} = \frac{7 - 6\beta}{15 - 6\beta}
\]
optimum $\kappa$ values

$$\kappa_{i-\frac{1}{2}} = \frac{7-6\beta}{9+6\beta}$$

$$\kappa_{i+\frac{3}{2}} = \frac{7-6\beta}{15-6\beta}$$

$\beta = \frac{1}{2}$: $\kappa_{i-\frac{1}{2}} = \kappa_{i+\frac{3}{2}} = \frac{1}{3}$
limited EB-affected cell-face states

\[ c_{i-\frac{1}{2}} = c_{i-1} + \frac{1}{2} \tilde{\phi}(\tilde{r}_{i-\frac{1}{2}})(c_{i-1} - c_{i-2}) \]

\[ \tilde{\phi}(\tilde{r}_{i-\frac{1}{2}}) = \frac{1+6\beta}{9+6\beta} + \frac{8}{9+6\beta} \tilde{r}_{i-\frac{1}{2}} \]

\[ \tilde{r}_{i-\frac{1}{2}} = \frac{2}{1+2\beta} \frac{c_{l} - c_{i-1}}{c_{i-1} - c_{i-2}} \]

\[ c_{i+\frac{3}{2}} = c_{i+1} + \frac{1}{3-2\beta} \tilde{\phi}(\tilde{r}_{i+\frac{3}{2}})(c_{i+1} - c_{EB}') \]

\[ \tilde{\phi}(\tilde{r}_{i+\frac{3}{2}}) = \frac{4}{15-6\beta} + \frac{11-6\beta}{15-6\beta} \tilde{r}_{i+\frac{3}{2}} \]

\[ \tilde{r}_{i+\frac{3}{2}} = \frac{3-2\beta}{2} \frac{c_{i+2} - c_{i+1}}{c_{i+1} - c_{EB}'} \]
limited EB-affected cell-face states

\[ c_{i-\frac{1}{2}} = c_{i-1} + \frac{1}{2} \phi(\tilde{r}_{i-\frac{1}{2}})(c_{i-1} - c_{i-2}) \]

\[ \phi(\tilde{r}_{i-\frac{1}{2}}) = \frac{1+6\beta}{9+6\beta} + \frac{8}{9+6\beta} \tilde{r}_{i-\frac{1}{2}} \]

\[ \tilde{r}_{i-\frac{1}{2}} = \frac{2}{1+2\beta} \frac{c_{EB}^l - c_{i-1}}{c_{i-1} - c_{i-2}} \]

\[ c_{i+\frac{3}{2}} = c_{i+1} + \frac{1}{3-2\beta} \phi(\tilde{r}_{i+\frac{3}{2}})(c_{i+1} - c_{EB}^r) \]

\[ \phi(\tilde{r}_{i+\frac{3}{2}}) = \frac{4}{15-6\beta} + \frac{11-6\beta}{15-6\beta} \tilde{r}_{i+\frac{3}{2}} \]

\[ \tilde{r}_{i+\frac{3}{2}} = \frac{3-2\beta}{2} \frac{c_{i+2} - c_{i+1}}{c_{i+1} - c_{EB}^r} \]

imposing monotonicity requirements:

\[ \frac{c_{i-\frac{1}{2}} - c_{i-\frac{3}{2}}}{c_{i-1} - c_{i-2}} \geq 0, \ldots \]

\[ \frac{\phi(\tilde{r}_{i-\frac{1}{2}})}{\tilde{r}_{i-\frac{1}{2}}} \leq 1 + 2\beta \]

\[ \frac{\phi(\tilde{r}_{i+\frac{3}{2}})}{\tilde{r}_{i+\frac{3}{2}}} \leq 2 \]
semi-discrete eqn: \[
\frac{dc_i}{dt} = -\frac{u}{h}(c_{i+\frac{1}{2}} - c_{i-\frac{1}{2}}) \equiv F(c)
\]
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\frac{dc_i}{dt} = -\frac{u}{h}(c_{i+\frac{1}{2}} - c_{i-\frac{1}{2}}) \equiv F(c)
\]

Forward Euler:
\[
c_{i}^{n+1} = c_{i}^{n} + \tau F(c^n)
\]
semi-discrete eqn: \[ \frac{dc_i}{dt} = -\frac{u}{h}(c_{i+\frac{1}{2}} - c_{i-\frac{1}{2}}) \equiv F(c) \]

Forward Euler:

\[ c_{i}^{n+1} = c_{i}^{n} + \tau F(c^n) \]

Modified Euler:

predict: \[ \hat{c}_{i}^{n+1} = c_{i}^{n} + \tau F(c^n) \]

correct: \[ c_{i}^{n+1} = c_{i}^{n} + \frac{\tau}{2} \left( F(c^n) + F(\hat{c}^{n+1}) \right) \]
Harten (1984) provided theorem that gives additional conditions necessary for convergence to monotone solutions.
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Theorem (Harten’s)

Consistent scheme written in conservation form

\[ c_{i}^{n+1} = c_{i}^{n} - D_{i-\frac{1}{2}}^{-} (c_{i}^{n} - c_{i-1}^{n}) + D_{i+\frac{1}{2}}^{+} (c_{i+1}^{n} - c_{i}^{n}) \]

is TVD if \( D_{i+\frac{1}{2}}^{\pm} \geq 0 \) and \( D_{i+\frac{1}{2}}^{-} + D_{i+\frac{1}{2}}^{+} \leq 1 \)
Harten (1984) provided theorem that gives additional conditions necessary for convergence to monotone solutions.

**Theorem (Harten’s)**

A consistent scheme written in conservation form

\[ c_{i}^{n+1} = c_{i}^{n} - D_{i-\frac{1}{2}}^{-} (c_{i}^{n} - c_{i-1}^{n}) + D_{i+\frac{1}{2}}^{+} (c_{i+1}^{n} - c_{i}^{n}) \]

is TVD if \( D_{i+\frac{1}{2}}^{\pm} \geq 0 \) and \( D_{i+\frac{1}{2}}^{-} + D_{i+\frac{1}{2}}^{+} \leq 1 \)

- Above conditions define upper bounds for limiter functions.
- Monotonicity yields more stringent restrictions on time step than stability.
imposing Harten’s conditions yields \( \nu = \frac{u \tau}{h} \) is CFL number:

\[
0 \leq \phi(\tilde{r}_{i-\frac{1}{2}}) \leq \frac{2}{\nu} - 2
\]

\[
-1 \leq \phi(\tilde{r}_{i+\frac{3}{2}}) \leq \frac{3-2\beta}{\nu} - 1
\]

\[
4 - \frac{2}{\nu} \leq \frac{\phi(\tilde{r}_{i-\frac{1}{2}})}{\tilde{r}_{i-\frac{1}{2}}} \leq 1 + 2\beta
\]

\[
4 - \frac{2}{\nu} \leq \frac{\phi(\tilde{r}_{i+\frac{3}{2}})}{\tilde{r}_{i+\frac{3}{2}}} \leq 2
\]
imposing Harten’s conditions yields \( \nu = \frac{u \tau}{h} \) is CFL number:

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0 \leq \tilde{\phi}(\tilde{r}_{i-\frac{1}{2}}) \leq \frac{2}{\nu} - 2
\]

\[
-1 \leq \tilde{\phi}(\tilde{r}_{i+\frac{3}{2}}) \leq \frac{3 - 2\beta}{\nu} - 1
\]

\[
4 - \frac{2}{\nu} \leq \frac{\tilde{\phi}(\tilde{r}_{i+\frac{1}{2}})}{\tilde{r}_{i+\frac{1}{2}}} \leq 2
\]

- EB-sensitive, \( \nu \)-dependent bounds
- fully-bound monotonicity domain
simplified monotonicity domains

\[ 0 \leq \tilde{\phi}(\tilde{r}_{i-\frac{1}{2}}) \leq 2 \]

\[ -1 \leq \tilde{\phi}(\tilde{r}_{i+\frac{3}{2}}) \leq 5 - 4\beta \]

\[ 0 \leq \frac{\tilde{\phi}(\tilde{r}_{i-\frac{1}{2}})}{\tilde{r}_{i-\frac{1}{2}}} \leq 1 + 2\beta \]

\[ 0 \leq \frac{\tilde{\phi}(\tilde{r}_{i+\frac{3}{2}})}{\tilde{r}_{i+\frac{3}{2}}} \leq 2 \]

- \( \nu \)-dependence avoided by taking stringent restriction on \( \nu \)
- monotonicity preserving schemes for \( \nu \leq \frac{1}{2} \)
Typical EB-sensitive limiters for the EB-affected fluxes, and their corresponding simplified monotonicity domains, for $\beta = \frac{1}{2}$
Flux integration techniques

set up: an EB crossing a cell interface
Flux integration techniques

\[ f_{i-\frac{1}{2}}^{n} \quad f_{i+\frac{3}{2}}^{n} \]

flux variations
Flux integration techniques

\[ f_{i+\frac{1}{2}}^{n+1} - f_{i+\frac{1}{2}}^n \]

flux variations: abrupt change in \( c_{i+\frac{1}{2}} \) at \( t^{n+\alpha} \)
Flux integration techniques

analytic integration: exact
 Flux integration techniques

numerical integration: Forward Euler
Flux integration techniques

numerical integration: Modified Euler
Adaptive Forward Euler: time-adapted $c_{i+\frac{1}{2}}$
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Adaptive Forward Euler: time-adapted $c_{i+\frac{1}{2}}$

\[ \alpha = \frac{x_{i+\frac{1}{2}} + \epsilon - x_E^p}{u\tau}, \quad \alpha \in (0, 1) \]
Adaptive Forward Euler: time-adapted $c_{i+\frac{1}{2}}$

\[ \alpha = \frac{x_{i+\frac{1}{2}} + \epsilon - x_{EB}^n}{u\tau}, \quad \alpha \in (0, 1) \]

\[ c_{i+\frac{1}{2}}^{n+1} := \alpha c_{i+\frac{1}{2}}^n + (1 - \alpha)c_{i+\frac{1}{2}}^{n+\alpha} \]
Adaptive Forward Euler: time-adapted $c_{i+\frac{1}{2}}$

\[ \alpha = \frac{x_{i+\frac{1}{2}} + \epsilon - x_{EB}^n}{u\tau}, \quad \alpha \in (0, 1) \]

\[ c_{i+\frac{1}{2}}^n := \alpha c_{i+\frac{1}{2}}^{n} + (1 - \alpha) c_{i+\frac{1}{2}}^{n+\alpha} \]

proceed time stepping with $\tau$
Standard Forward Euler: no time-splitting for discontinuous flux
Adaptive Forward Euler: with *time-splitting* for discontinuous flux
Adaptive Modified Euler: time-adapted predicted/corrected cell-face states

\[ t^{n+1} \]

\[ X \]

\[ n \]

\[ n+1 \]
Adaptive Modified Euler: time-adapted predicted/corrected cell-face states

compute $\alpha^-$
Adaptive Modified Euler: time-adapted predicted/corrected cell-face states

compute $\alpha^-$, predict solutions and capture $\hat{c}_{i+\frac{1}{2}}^{n+\alpha^-}$
Adaptive Modified Euler: time-adapted predicted/corrected cell-face states

- compute $\alpha^-$, predict solutions and capture $\hat{c}_{i+\frac{1}{2}}^{n+\alpha^-}$
- compute $\alpha^+$
Adaptive Modified Euler: time-adapted predicted/corrected cell-face states

- compute $\alpha^-$, predict solutions and capture $\hat{c}^{n+\alpha^-}_{i+\frac{1}{2}}$
- compute $\alpha^+$, correct solutions and capture $c^{n+\alpha^+}_{i+\frac{1}{2}}$
Adaptive Modified Euler: time-adapted predicted/corrected cell-face states

- compute $\alpha^-$, predict solutions and capture $\hat{c}_{i+\frac{1}{2}}^{n+\alpha^-}$
- compute $\alpha^+$, correct solutions and capture $c_{i+\frac{1}{2}}^{n+\alpha^+}$
- move EB to its ‘final destination’
Adaptive Modified Euler: time-adapted predicted/corrected cell-face states

- Compute $\alpha^-$, predict solutions and capture $\hat{c}^{n+\alpha^-}_{i+\frac{1}{2}}$
- Compute $\alpha^+$, correct solutions and capture $c^{n+\alpha^+}_{i+\frac{1}{2}}$
- Move EB to its ‘final destination’, predict solutions and capture $\hat{c}^{n+1}_{i+\frac{1}{2}}$
Standard Modified Euler: no time-splitting for discontinuous flux
Adaptive Modified Euler: with time-splitting for discontinuous flux
with $\nu \sim O(10^{-3})$

exact discrete ($\bigcirc \cdots$), unlimited high-order upwind-biased ($\blacksquare -$), limited ditto ($\ast -$)
with $\nu \sim \mathcal{O}(10^{-3})$

exact discrete (○ ⋯), unlimited high-order upwind-biased (□ −), limited ditto (∗ −)
Adaptive Forward Euler: $\nu = 0.15, T = 1$
Adaptive Modified Euler: $\nu = 0.75$, $T = 1$
Adaptive Modified Euler: $\nu = 0.75, \ T = 5$
Moving boundaries accurately and efficiently embedded in uniform fixed-grid

- Tailor-made limiters derived for EB-affected fluxes
- Monotone, high-order accurate (in space and time) solution achieved
- Remarkably accurate results without much computational overhead
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